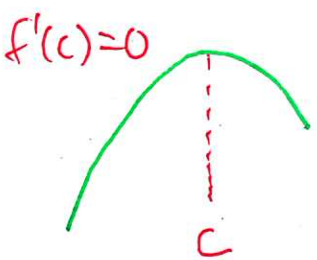
# Week 7

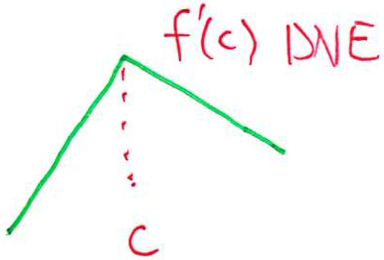
## First derivative test

Let $f$ be continuous at $c \in (a,b)$

(i) If $f'(x) > 0$ on $(a,c)$, $f'(x) < 0$ on $(c,b)$ then $f$ has a relative maximum at $c$

Rmk $f$ may not be differentiable at $c$

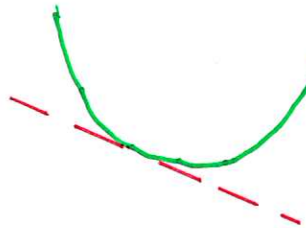


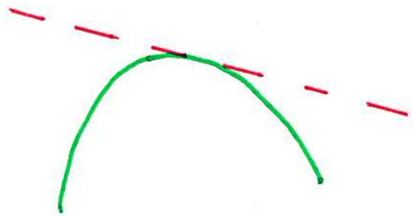

(ii) If $f'(x) < 0$ on $(a,c)$, $f'(x) > 0$ on $(c,b)$ then $f$ has a relative minimum at $c$

## Concavity

Concave up



Concave down



Graph lies above tangent

Graph lies below tangent

Slope $= f'$ is increasing

Slope $= f'$ is decreasing

Def A point of inflection is where $f$ changes concavity

Prop Let $I$ be an interval

$f$ is twice differentiable on $I$

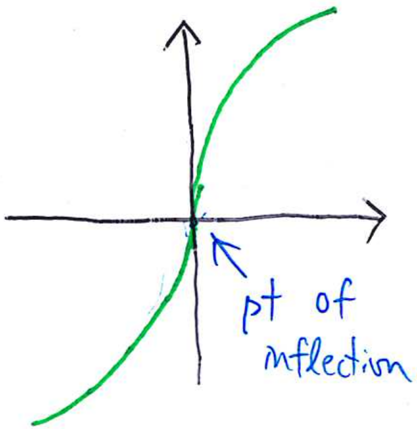
ie. $f''(x)$ exists $\forall x \in I$

i. If $f''(x) > 0$ on $I$, then $f$ is concave up on $I$

ii. If $f''(x) < 0$ on $I$, then $f$ is concave down on $I$

eg $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$

$f(x) = x^{\frac{1}{3}}$

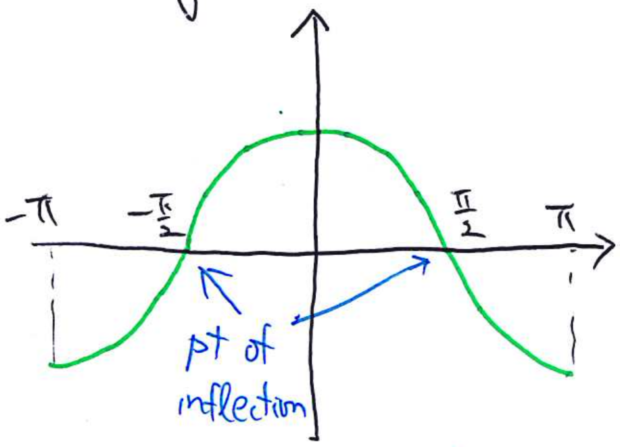



Concave up — Concave down

$f'' > 0$ — $f'' < 0$

$g(x) = \cos x$

$g''(x) = -\cos x$




concave down

concave up

## Second derivative test

Suppose $f'(c) = 0$. If $f''(c) > 0$ ($f''(c) < 0$) then $f$ has relative minimum (maximum) at $c$

No conclusion if $f'(c) = 0$

$f'' > 0$ — $f'' < 0$

## Optimization

Let $f : [a,b] \to \mathbb{R}$ be continuous,

Extreme Value Thm $\Rightarrow$ $f$ has absolute max and min

**Q** How to find them?

**Fact:** If $f$ has an extremum at $c \in [a,b]$,

Then
1. $f'(c) = 0$ or
2. $f'(c)$ DNE
3. $c$ is an endpoint, i.e. $c = a$ or $b$

$c$ is called a critical point

**Strategy:** Find critical points and endpoints

and compare values of $f$ at those points

**eg** $f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$

Find the absolute max and min of $f$ on $[-1,1]$

**Sol** Note that $f$ is continuous on $[-1,1]$

EVT $\Rightarrow$ Absolute max and min exist

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}} \quad \text{for } x \neq 0$$

Note that $f$ is not differentiable at $0$:

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{5}{3}}+2h^{\frac{2}{3}}}{h} = \lim_{h\to 0} \underbrace{h^{\frac{2}{3}}}_{\to 0} + \underbrace{2h^{-\frac{1}{3}}}_{\text{DNE}}$$

$$f'(x) = 0 \iff \frac{5}{3}x^{\frac{2}{3}} + \frac{4}{3}x^{-\frac{1}{3}} = 0$$

$$\iff \frac{x^{-\frac{1}{3}}}{3}(5x+4) = 0$$

$$\iff x = -\frac{4}{5}$$

Two critical points: $0, -\frac{4}{5}$

end points: $-1, 1$

$f(-1) = -1 + 2 = 1$

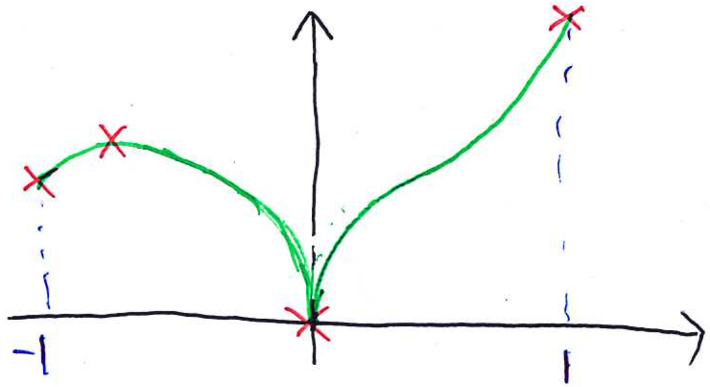
$f(1) = 1 + 2 = 3$

$f(0) = 0$

$f(-\frac{4}{5}) = (-\frac{4}{5})^{\frac{5}{3}} + 2(-\frac{4}{5})^{\frac{2}{3}} \approx 1.034$

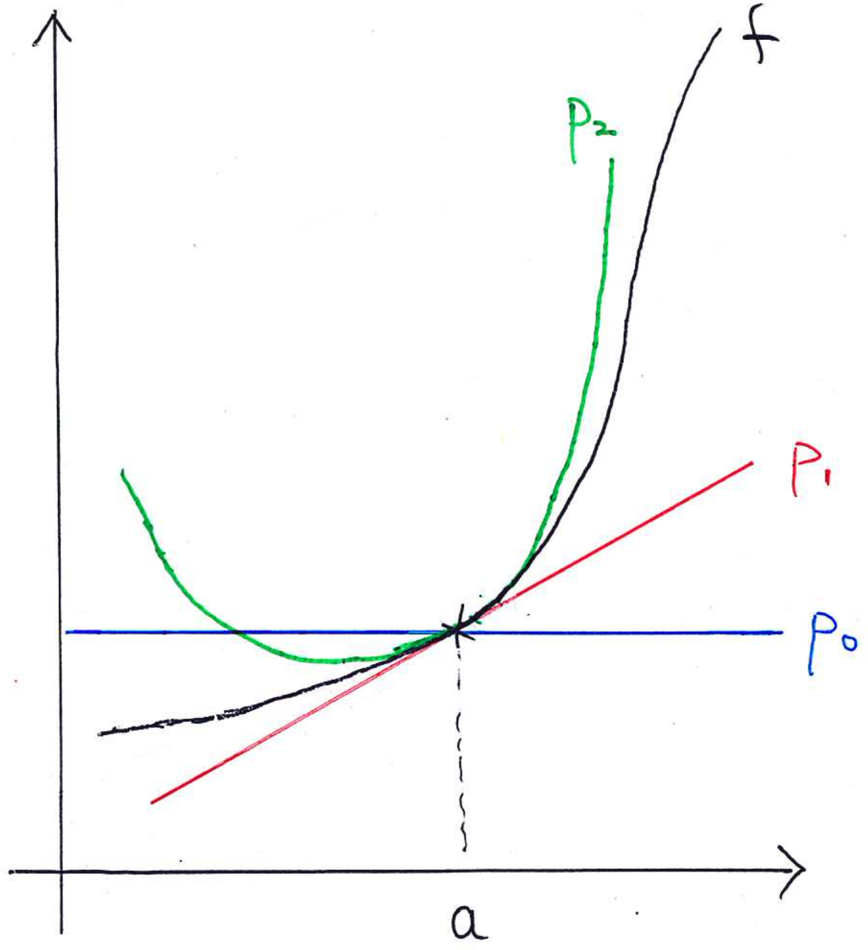
Comparison $\Rightarrow$ max at $1$, $f(1) = 3$

min at $0$ $f(0) = 0$



# Taylor Polynomial

Approximate a function $f$ by a polynomial $p_n$ of degree $\leq n$ near $a$




$\underline{n=0}$ $p_0(x)$ is a constant

Take $p_0(x) = f(a)$ $p_0, f$ have same value at $a$

$\underline{n=1}$ $p_1(x)$ is a linear polynomial

Take $p_1(x) = f(a) + \underbrace{f'(a)}_{\text{slope}}(x-a)$

$p_1, f$ have same value and slope at $a$

$\underline{n=2}$ $p_2(x)$ is a quadratic polynomial

Take $p_2(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2$

Note that $p_2'(x) = f'(a) + f''(a)(x-a)$

$p_2''(x) = f''(a)$

$p_2(a) = f(a)$ $p_2'(a) = f'(a)$ $p_2''(a) = f''(a)$

$\Rightarrow$ same value, slope, concavity at $a$

**Defn** Let $f$ be a n-time differentiable function at $a$

i.e. $f^{(k)}(a)$ exists for $0 \le k \le n$ $\quad (f^{(0)} = f)$

Define the $\boxed{\text{n-th order Taylor polynomial at a}}$ to be

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (\deg \le n)$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

**Rmk:** $0! = 1 \qquad n! = (n-1)! \cdot n$

**Ex** Check $\boxed{P_n^{(k)}(a) = f^{(k)}(a) \text{ for } 0 \le k \le n}$

**Rmk** $P_n(x)$ is the "best" polynomial of degree $\le n$ to approximate $f$ near $a$

eg Let $f(x) = \cos x$

1. Find Taylor polynomial of $f$ at $0$

2. Approximate $f(0.1) = \cos(0.1)$ using $p_0, p_2, p_4$.

Sol

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x \Rightarrow f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k = 4m, m \in \mathbb{Z} \\ -1 & \text{if } k = 4m+2 \end{cases}$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

same

$$\begin{matrix} P_{2n}(x) \\ P_{2n+1}(x) \end{matrix} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$

$p_0 = p_1 = 1$

$P_2(x) = P_3(x) = 1 - \frac{1}{2!}x^2$

$P_4(x) = P_5(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

Approximation: $p_0(0.1) = 1$

$P_2(0.1) = 0.995$

$P_4(0.1) = 0.995004166\cdots$

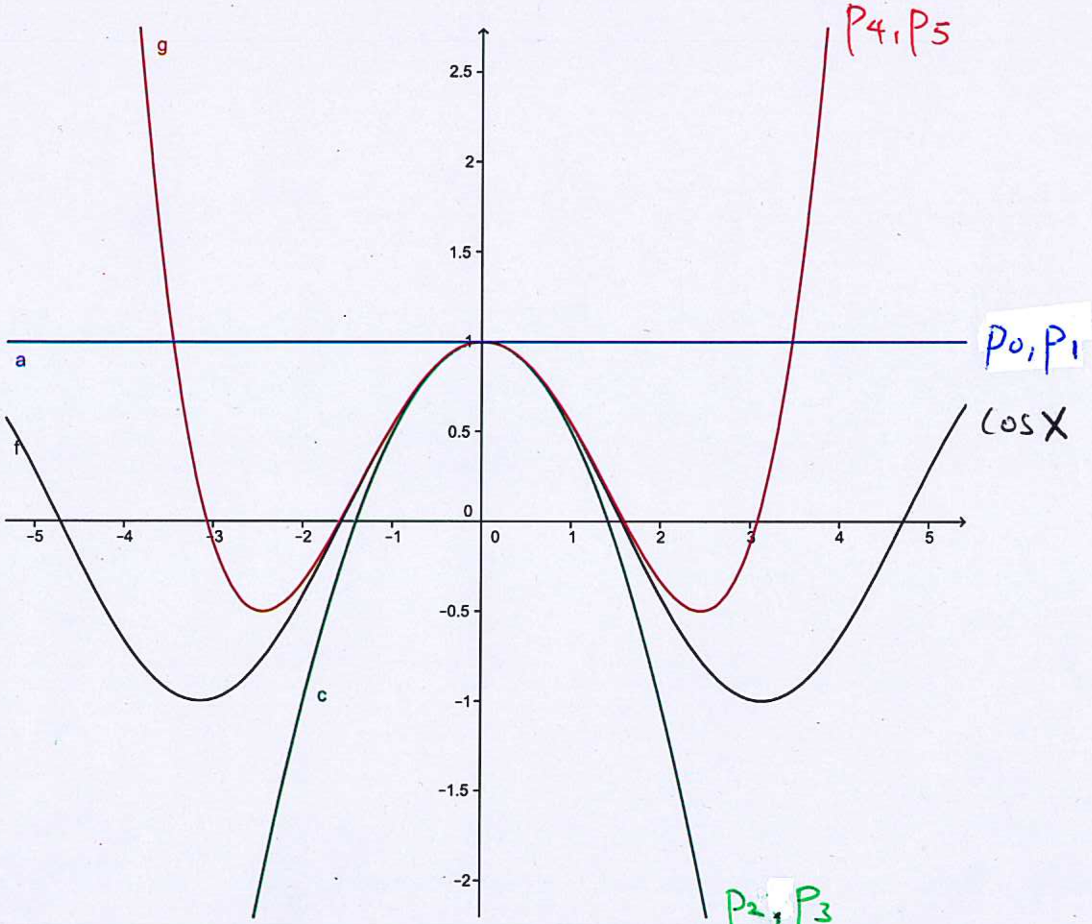
Actual value:

$f(0.1) = \cos(0.1)$

$= 0.99500416527\cdots$

$f(x) = \cos x$

$P_n$ = n-th order Taylor polynomial at 0




Larger $n \Rightarrow P_n$ is a better approximation when $x$ is close enough to $a$ ($a=0$ in this example)

$\underline{\text{eg}}$ Find Taylor polynomial for $f(x) = \ln x$ at $a=1$

$\underline{\text{Sol}}$ $f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} \qquad f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{3 \cdot 2}{x^4}$

For $n \geq 1$, $f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}$

$$\therefore f^{(n)}(1) = \begin{cases} \ln 1 = 0 & \text{if } n=0 \\ (-1)^{n+1}(n-1)! & \text{if } n \geq 1 \end{cases}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-a)^k$$

1st term = 0

$$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

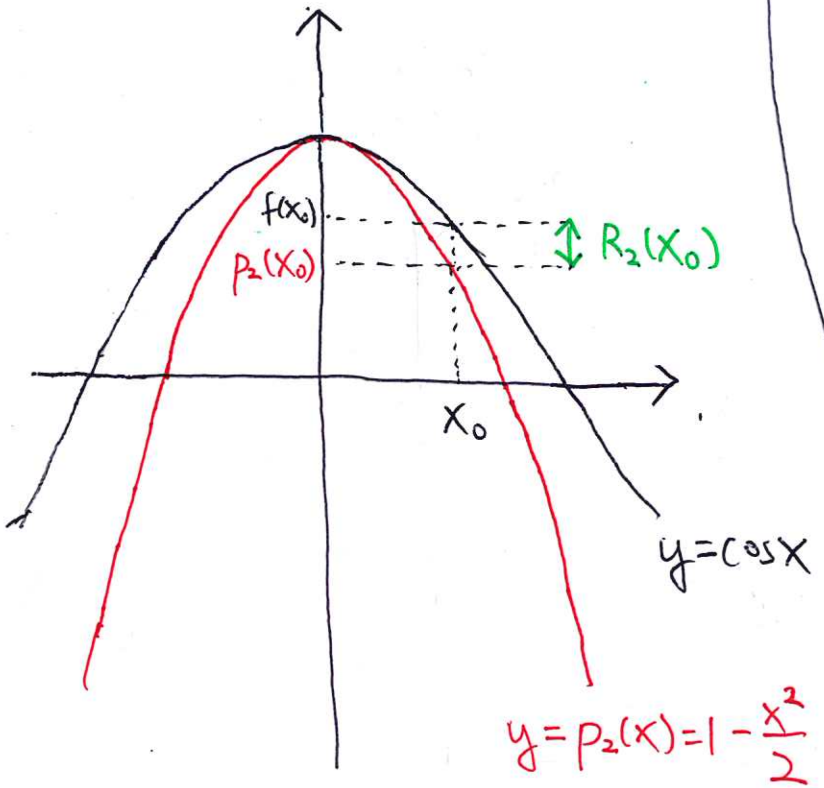
Approximation: $f(x) \approx p_n(x)$

Q: How accurate is it?

$\Rightarrow$ Study $R_n(x) = f(x) - p_n(x)$

Error / Remainder

$$\boxed{f(x) = p_n(x) + R_n(x)}$$




**Taylor's theorem** Let $x \neq a$, i.e. $a < x$ or $x < a$

Suppose $f^{(n)}$ exists and is continuous on $[a, x]$ (or $[x, a]$)

and $f^{(n+1)}$ exists on $(a, x)$ (or $(x, a)$)

Then $\exists\, c \in (a, x)$ (or $(x, a)$) such that

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{p_n(x),\ \text{where } p_n \text{ is the n-th order Taylor polynomial of } f \text{ at } a} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)}$$

i.e. $f(x) = p_n(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{called } \boxed{\text{Remainder in Lagrange form}}}$ for some $c$ between $x$ and $a$

Rmk ① The assumption means

$f', f'', \cdots, f^{(n-1)}$ exist and are differentiable on $[a, x]$ or $[x, a]$

② For $n = 0$, $\xRightarrow{\text{Theorem}} f(x) = f(a) + f'(c)(x-a) \Rightarrow \frac{f(x) - f(a)}{x - a} = f'(c)$

$\therefore$ Lagrange's MVT is a special case of Taylor's theorem

# Pf of Taylor's Theorem

Assume $x > a$.

Let $F(y) = f(y) - P_n(y) - \dfrac{f(x) - P_n(x)}{(x-a)^{n+1}}(y-a)^{n+1}$

x is regarded as a constant here

Use y instead of x to avoid confusion

same derivative up to order n at a

Zero derivative up to order n at a

## Idea of Pf (similar to that of MVT)

Repeated application of Rolle's theorem

① Check: $F$ is continuous on $[a, x]$

$F$ is differentiable on $(a, x)$

$F(a) = F(x)$ (Both are 0)

Rolle's Thm

$\Rightarrow \exists c_1 \in (a, x)$ such that $F'(c_1) = 0$

② Apply Rolle's thm to $F'$ on $(a, c_1)$

Check: $F'$ is continuous on $[a, c_1]$

$F'$ is differentiable on $(a, c_1)$

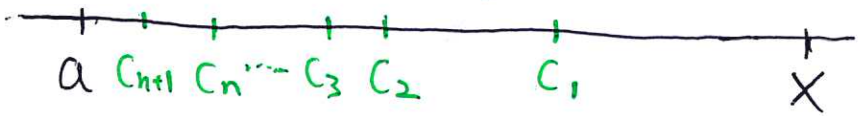
$F'(a) = 0 \underset{\text{step 1}}{=} F'(c_1)$

Rolle's thm

$\Rightarrow \exists c_2 \in (a, c_1)$ such that $F''(c_2) = 0$

③ Applying Rolle's thm to $F'', F''', \cdots F^{(n)}$ similarly

$\Rightarrow \exists c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c_{n+1}) = 0$




Since $F^{(n+1)}(y) = f^{(n+1)}(y) - (n+1)!\dfrac{f(x) - P_n(x)}{(x-a)^{n+1}}$

$F^{(n+1)}(c_{n+1}) = 0 \Rightarrow f^{(n+1)}(c_{n+1}) = (n+1)!\dfrac{f(x) - P_n(x)}{(x-a)^{n+1}}$

4 Take $c = c_{n+1}$

We will verify that $c$ satisfies the statement of the theorem

$$p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n-1}$$

$$\overset{3}{=} p_n(x) + f(x) - p_n(x)$$

$$= f(x)$$

$\Rightarrow$ Taylor's theorem

(The proof of the case $x<a$ is similar)